Boninger's Mistake

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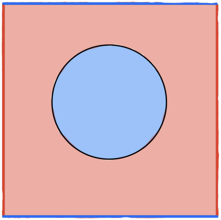
Checkerboard Colorablitity of a Surface

Definition 1

Let Σ be a closed orientable surface and D be a link diagram of a link $L \subset \Sigma \times [0,1]$. We say that the diagram D is Checkerboard colorable if each connected region of $\Sigma \backslash D$ can be colored by $\{0,1\}$ such that there is no arc of D whach have 2 sides that are the same color.



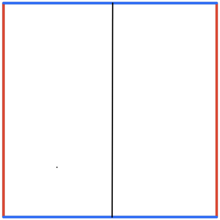


Figure: Checkerboard Colorable and Non-Checkerboard Colorable Embeddings of the Unknot in $\mathbb{T}^2$

Thistlethwaite's Polynomial

Definition 2

In [TH], Thistlethwaite defined the Laurent polynomial $\tau[\Gamma]$ of a Graph Γ recursively by;

$\tau[\Gamma] = A_e^{-1}\tau[\Gamma - e] + A_e\tau[\Gamma/e]$	If e is not a bridge nor loop;
$\tau[\Gamma] = A_e^{-3}\tau[\Gamma/e]$	If e is a bridge;
$\tau[\Gamma] = A_e^{3}\tau[\Gamma - e]$	If e is a loop;
$\tau[\Gamma_1 \sqcup \Gamma_2] = d\tau[\Gamma_1]\tau[\Gamma_2]$	For disjoint union $\Gamma_1 \sqcup \Gamma_2$;
$\tau[\cdot] = 1$	

Where $d = -A^2 - A^{-2}$ and $A_e = \begin{cases} A & \text{if } \mathrm{sign}(e) = + \\ A^{-1} & \text{if } \mathrm{sign}(e) = - \end{cases}$

Boninger's Polynomial from a Graph

Definition 3

Let D be a checkerboard colorable Virtual link diagram and Γ be the associated Signed Tait Graph then we can define;

$$\nu_{D,\Gamma}(t) = \left((-A)^{-3w(D)} \tau[\Gamma] \right)_{A^{-2}=t^{1/2}}$$

The Theorem of Boninger

The following was a result given in [BO]:

Theorem 4 (Boninger 23')

Let Σ be a closed orientable surface. Let $D \subset \Sigma$ be a checkerboard colorable, non-split link diagram, and $L \subset \Sigma \times [0,1]$ the associated link. Let Γ and Γ' be the signed Tait graphs associated to the two checkerboard colorings of D. Then,

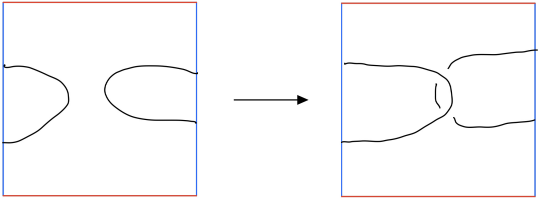
$$\{\nu_{D,\Gamma}(t), \nu_{D,\Gamma'}(t)\}$$

Is an isotopy invariant of L.

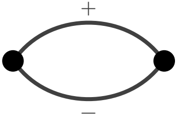
This Theorem turns out to be false as it is currently stated.

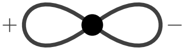
Counter-Example

Consider the following Knot on $\mathbb{T}^2$



The diagram on the right, which we will call D has Tait graphs

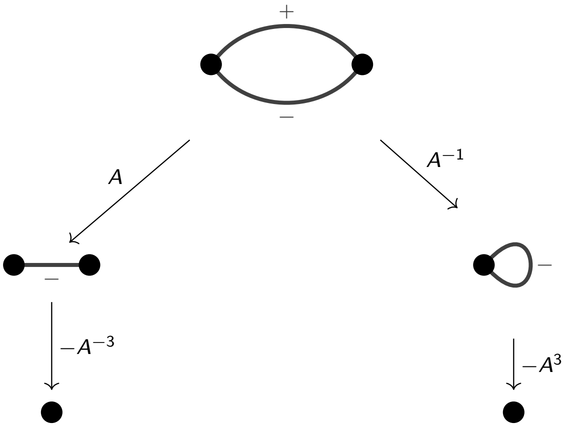




We will call this graph Γ We will call this graph Γ'

Counter-Example

We can easily see that $\tau[\Gamma'] = 1$. We can compute,



Counter-Example

This means $\tau[\Gamma] = -A^2 - A^{-2}$. Since $w(D) = 0$ the set that Boninger's theorem claims is an invariant is,

$$\{1, -A^2 - A^{-2}\}$$

Notice that D is a diagram for the unknot, so we can campute from the other diagram given that the set Boninger's Theorem calims is invariant is,

$$\{1, 1\}$$

This gives us a Counter-Example to Boninger's theorem.

Essential Annulus

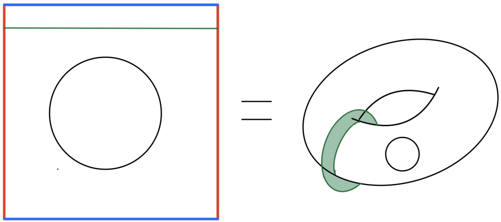
Definition 5

Let Σ be a closed compact orientable surface and $L \subset \Sigma \times [0,1]$ be a Link on Σ. We say that $A \subset (\Sigma \times [0,1]) \backslash L$ is an Essential Annulus if,

- A is homeomorphic to the Annulus in the plane which we will denote $\mathbb{A} = \{x \in \mathbb{R}^2 : 1 \leqslant |x| \leqslant 2\}$
- One boundary component of A which is denoted ∂A_1 is contained in $(\Sigma \times \{1\}) \backslash L$. Additionally the other boundary component of A which is denoted ∂A_2 is contained in $(\Sigma \times \{0\}) \backslash L$
- A does not bound a ball in $(\Sigma \times [0,1]) \backslash L$. Analagously this means that in $(\Sigma \times [0,1]) \backslash L$ it is not possible to continuously shrink A to a line where ∂A_1 stays contained in $(\Sigma \times \{1\}) \backslash L$ and ∂A_2 stays contained in $(\Sigma \times \{0\}) \backslash L$

Example of Essential Annulus

For our Counter-Example knot we have an Essential Annulus. To illustrate this, note that because of the topological nature of the definition of Essential Annulus it is a property of the link L which means it doesenn't depend on the diagram D of the link. So consider the following diagram of the unknot of $\mathbb{T}^2$,



Observe that the green surface is an essential Annulus.

A Possible Fix

The following was reccomended as a possible fix to the problem in an email correspondence with Dr. Boninger.

Theorem 6

Let Σ be a closed orientable surface. Let $L \subset \Sigma \times [0,1]$ be a Link with no Essential Annulus on $\Sigma \times [0,1]$ and $D \subset \Sigma$ be a checkerboard colorable, non-split link diagram of L. If Γ and Γ' are the signed Tait graphs associated to the two checkerboard colorings of D then,

$$\{\nu_{D,\Gamma}(t), \nu_{D,\Gamma'}(t)\}$$

Is an isotopy invariant of L.

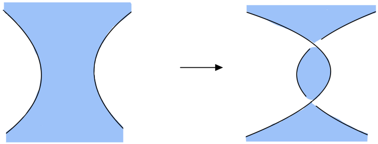
Why Should This Fix Boninger's Theorem

Note that the counter-example we found shows that his proof for Reidemeister 2 was wrong. In fact Boninger's proofs for Reidemeister 2 and 3 were both wrong, but it can be shown that the original statement was still true for Reidemeister 3. The Issue with the given proof is Boninger assumes that some regions of the checkerboard coloring are distinct.

If we assume that all of the regions we consider under the Reidemeister moves are distinct then the proof given by Boninger holds true.

Proving Invariance Under Reidemeister 2

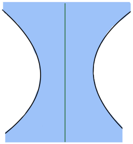
Consider the following checkerboard coloring for a link L on the surface Σ described in the fixed version of the Theorem.



If the upper region and lower region on the right diagram are the same region we have that there is a path from the upper region to the lower region that does not pass through any crossings in the right diagram.

Proving Invariance Under Reidemeister 2

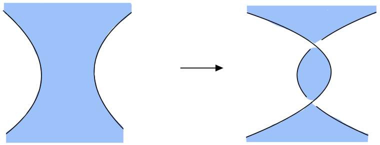
Since we have this path we can construct the following surface on $\Sigma \times [0,1]$.



We can notice that the green line drawn in this diagram represents a surface that we can see is clarly homeomorphic to an annulus because the top of the line connects back to the bottom of the line as depicted. We can note that since we have strands of the knot on both sides of the annulus it does not bound a ball in $(\Sigma \times [0,1]) \backslash L$. Thus the surface drawn is an essential annulus.

Proving Invariance Under Reidemeister 2

So we obtain a contradiction.



This means that on the right diagram we must have that the upper region is not the same region as the lower region. Thus we can proceed to use the proof Boninger gave to show that with this additional assumption we have that the set of polynomials $\{\nu_{D,\Gamma}(t), \nu_{D,\Gamma'}(t)\}$ is invariant under the Reidemeister 2 move.

References

GK Greg Kuperberg, *What is a virtual link?*, Algebraic & Geometric Topology, Volume 3 (2003) 587–591; preprint https://arxiv.org/pdf/math/0208039.

TH Morwen B. Thistlethwaite, A spanning tree expansion of the Jones polynomial, Topology 26 (1987), no. 3, 297–309

BO Boninger, J. (2022). The Jones Polynomial from a Goeritz Matrix. Bulletin of the London Mathematical Society, 55(2), 732-755. https://doi.org/10.1112/blms.12753

KA L.H. Kauffman, A Tutte polynomial for signed graphs, Discrete Appl. Math. 25 (1989) 105–127.